PERIODIC AND ASYMPTOTICALLY PERIODIC SOLUTIONS OF SYSTEMS OF NONLINEAR DIFFERENCE EQUATIONS WITH INFINITE DELAY

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Abstract. In this work we study the existence of periodic and asymptotically periodic solutions of a system of nonlinear Volterra difference equations with infinite delay. By means of fixed point theory, we furnish conditions that guarantee the existence of such periodic solutions.

1. Introduction

Consider the system of nonlinear Volterra difference equations with infinite delay

$$\begin{cases} \Delta x_n = h_n x_n + \sum\limits_{i=-\infty}^{n} a_{n,i} f(y_i) \\ \Delta y_n = p_n y_n + \sum\limits_{i=-\infty}^{n} b_{n,i} g(x_i) \end{cases}, \tag{1.1}$$

where f and g are real valued and continuous functions, and $\{a_{n,i}\}$, $\{b_{n,i}\}$, $\{h_n\}$, and $\{p_n\}$ are real sequences. In this study, we use *Schauder's fixed point theorem* to provide sufficient conditions guaranteeing the existence of periodic and asymptotically periodic solutions of the system (1.1). Since we are seeking the existence of periodic solutions it is natural to ask that there exists a least positive integer T such that

$$h_{n+T} = h_n,\ p_{n+T} = p_n, \tag{1.2}$$

$$a_{n+T,i+T} = a_{n,i}, \tag{1.3}$$

and

$$b_{n+T,i+T} = b_{n,i} \tag{1.4}$$

hold for all $n \in \mathbb{N}$, where $\mathbb{N}$ indicates the set of all nonnegative integers.

Recently, there has been a remarkable interest in the study of Volterra equations due to their applications in numerical analysis and biological systems, see e.g. [1], [3], and [19]. There is a vast literature on this subject in the continuous and discrete cases. For instance, in [5] the authors considered the two dimensional system of nonlinear Volterra difference equations

$$\begin{cases} \Delta x_n = h_n x_n + \sum\limits_{i=1}^{n} a_{n,i} f(y_i) \\ \Delta y_n = p_n y_n + \sum\limits_{i=1}^{n} b_{n,i} g(x_i) \end{cases}, \quad n = 1, 2, ...$$

and classified the limiting behavior and the existence of its positive solutions with the help of fixed point theory. Also, the authors of [17] analyzed the asymptotic behavior of positive solutions of

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second order nonlinear difference systems, while the authors of [18] studied the classification and the existence of positive solutions of the system of Volterra nonlinear difference equations. Periodicity of the solutions of difference equations has been handled by [2], [6]-[11]. In [7] and [8], the authors focused on a system of Volterra difference equations of the form

$$x_s(n) = a_s(n) + b_s(n)x_s(n) + \sum_{p=1}^{r}\sum_{i=0}^{n} K_{sp}(n,i)x_p(i), \quad n \in \mathbb{N},$$

where a_s, b_s, $x_s : \mathbb{N} \to \mathbb{R}$ and $K_{sp} : \mathbb{N}\times\mathbb{N} \to \mathbb{R}$, $s = 1, 2, ..., r$, and $\mathbb{R}$ denotes the set of all real numbers and obtained sufficient conditions for the existence of asymptotically periodic solutions. They had to construct a mapping on an appropriate space and then obtain a fixed point. Furthermore, in [12] the authors investigated the existence of periodic and positive periodic solutions of system of nonlinear Volterra integro-differential equations. The paper [10] of Elaydi, was one of the first to address the existence of periodic solutions and the stability analysis of Volterra difference equations. Since then, the study of Volterra difference equations has been vastly increasing. For instance, we mention the papers [13], [15], and the references therein. In addition to periodicity we refer to [14] and [16] for results regarding boundedness.
The main purpose of this paper is to extend the results of the above mentioned literature by investigating the possibility of existence of periodic and the asymptotic periodic solutions for systems of nonlinear Volterra difference equations with infinite delay.

Denote by $\mathbb{Z}$ and $\mathbb{Z}^-$ the set of integers and the set of nonpositive integers, respectively. By a solution of the system (1.1) we mean a pair of sequences $\{(x_n, y_n)\}_{n\in\mathbb{Z}}$ of real numbers which satisfies (1.1) for all $n \in \mathbb{N}$. The initial sequence space for the solutions of the system (1.1) can be constructed as follows. Let S denote the nonempty set of pairs of all sequences $(\eta, \zeta) = \{(\eta_n, \zeta_n)\}_{n\in\mathbb{Z}^-}$ of real numbers such that

$$\max\left\{\sup_{n\in\mathbb{Z}^-} |\eta_n|, \sup_{n\in\mathbb{Z}^-} |\zeta_n|\right\} < \infty,$$

and for each $n \in \mathbb{N}$ the series

$$\sum_{i=-\infty}^{0} a_{n,i} f(\eta_i) \text{ and } \sum_{i=-\infty}^{0} b_{n,i} g(\zeta_i)$$

converge. It is clear that for any given pair of initial sequences $\{(\eta_n, \zeta_n)\}_{n\in\mathbb{Z}^-}$ in S there exists a unique solution $\{(x_n, y_n)\}_{n\in\mathbb{Z}}$ of the system (1.1) which satisfies the initial condition

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix} \text{ for } n \in \mathbb{Z}^-, \tag{1.5}$$

such solution $\{(x_n, y_n)\}_{n\in\mathbb{Z}}$ is said to be the solution of the initial problem (1.1-1.5). For any pair $(\eta, \zeta) \in S$, one can specify a solution of (1.1-1.5) by denoting it by $\{(x_n(\eta), y_n(\zeta))\}_{n\in\mathbb{Z}}$, where

$$(x_n(\eta), y_n(\zeta)) = \begin{cases} (\eta_n, \zeta_n) & \text{for } n \in \mathbb{Z}^- \\ (x_n, y_n) & \text{for } n \in \mathbb{N} \end{cases}.$$

In our analysis, we apply a fixed point theorem to general operators over a Banach space of bounded sequences defined on the whole set of integers. Unlike the above mentioned literature that dealt with stability of delayed difference systems, in the construction of our existence type theorems we neglect the consideration of phase space, for simplicity. For a similar approach we refer to [4].

We end this section by recalling the fixed point theorem that we use in our further analysis.

Theorem 1. (Schauder's fixed point theorem)*Let X be a Banach Space. Assume that K is a closed, bounded and convex subset of X. If $T : K \to K$ is a compact operator, then it has a fixed point in K.*

2. Periodicity

In this section, we use Schauder's fixed point theorem to show that system (1.1) has a periodic solution.

Let P_T be the set of all pairs of sequences $(x, y) = \{(x_n, y_n)\}_{n\in\mathbb{Z}}$ satisfying $x_{n+T} = x_n$ and $y_{n+T} = y_n$ for all $n \in \mathbb{N}$. Then P_T is a Banach space when it is endowed with the maximum norm

$$\|(x, y)\| := \max\left\{\max_{n\in[1,T]_{\mathbb{Z}}} |x_n|\,, \max_{n\in[1,T]_{\mathbb{Z}}} |y_n|\right\},$$

where $[1, T]_{\mathbb{Z}} := [1, T] \cap \mathbb{Z}$. Let us define the subset $\Omega\,(W)$ of P_T by

$$\Omega\,(W) := \{(x, y) \in P_T : \|(x, y)\| \le W\},$$

where $W > 0$ is a constant. Then $\Omega\,(W)$ is a bounded, closed and convex subset of P_T.

For any pair $(x, y) = \{(x_n\,(\eta)\,, y_n\,(\zeta))\}_{n\in\mathbb{Z}} \in \Omega\,(W)$ with an initial sequence $\{(\eta_n, \zeta_n)\}_{n\in\mathbb{Z}^-}$ in S, define the mapping E on $\Omega\,(W)$ by

$$E\,(x, y) := \{E(x, y)_n\}_{n\in\mathbb{Z}} := \left\{\begin{pmatrix} E_1(x, y)_n \\ E_2(x, y)_n \end{pmatrix}\right\}_{n\in\mathbb{Z}},$$

where

$$E_1(x, y)_n := \begin{cases} \eta_n & \text{for } n \in \mathbb{Z}^- \\ \alpha_h \sum\limits_{i=n}^{n+T-1} \left(\prod\limits_{l=i+1}^{n+T-1} (1 + h_l)\right) \sum\limits_{m=-\infty}^{i} a_{i,m} f(y_m) & \text{for } n \in \mathbb{N} \end{cases}, \tag{2.1}$$

$$E_2(x, y)_n := \begin{cases} \zeta_n & \text{for } n \in \mathbb{Z}^- \\ \alpha_p \sum\limits_{i=n}^{n+T-1} \left(\prod\limits_{l=i+1}^{n+T-1} (1 + p_l)\right) \sum\limits_{m=-\infty}^{i} b_{i,m} g(x_m) & \text{for } n \in \mathbb{N} \end{cases}, \tag{2.2}$$

and

$$\alpha_h := \left[1 - \prod_{l=0}^{T-1} (1 + h_l)\right]^{-1},$$

$$\alpha_p := \left[1 - \prod_{l=0}^{T-1} (1 + p_l)\right]^{-1}.$$

We shall use the following result on several occasions in our further analysis.

Lemma 1. *Assume that (1.2-1.4) hold. Suppose that $1 + h_n \neq 0$, $1 + p_n \neq 0$ for all $n \in [1, T]_{\mathbb{Z}}$, and that*

$$\prod_{l=0}^{T-1} (1 + h_l) \neq 1 \ \textit{and} \ \prod_{l=0}^{T-1} (1 + p_l) \neq 1. \tag{2.3}$$

The pair $(x, y) = \{(x_n\,(\eta)\,, y_n\,(\zeta))\}_{n\in\mathbb{Z}} \in \Omega\,(W)$ with an initial sequence $\{(\eta_n, \zeta_n)\}_{n\in\mathbb{Z}^-}$ in S satisfies

$$E(x, y)_n = (x_n, y_n)$$

for all $n \in \mathbb{N}$ if and only if it is a T-periodic solution of (1.1).

Proof. One may easily verify that the pair $(x, y) = \{(x_n\,(\eta)\,, y_n\,(\zeta))\}_{n\in\mathbb{Z}} \in \Omega\,(W)$ satisfying $(x_n, y_n) = E(x, y)_n$ for all $n \in \mathbb{N}$ is a T-periodic solution of the system (1.1). Conversely, suppose that the pair

$(x,y)=\{(x_n(\eta),y_n(\zeta))\}_{n\in\mathbb{Z}}\in\Omega(W)$ is a T–periodic solution of (1.1). Multiplying both sides of the first equation in (1.1) with $\left(\prod_{l=0}^{n}(1+h_l)\right)^{-1}$ and taking the summation from n to $n+T-1$, we obtain

$$\sum_{i=n}^{n+T-1}\Delta\left[x_i\left(\prod_{l=0}^{i-1}(1+h_l)\right)^{-1}\right]=\sum_{i=n}^{n+T-1}\left(\prod_{l=0}^{i}(1+h_l)\right)^{-1}\sum_{m=-\infty}^{i}a_{i,m}f(y_m).$$

This implies that

$$x_{n+T}\left(\prod_{l=0}^{n+T-1}(1+h_l)\right)^{-1}-x_n\left(\prod_{l=0}^{n-1}(1+h_l)\right)^{-1}$$
$$=\sum_{i=n}^{n+T-1}\left(\prod_{l=0}^{i}(1+h_l)\right)^{-1}\sum_{m=-\infty}^{i}a_{i,m}f(y_m).$$

Using the equalities $x_{n+T}=x_n$ and $\prod_{l=n}^{n+T-1}(1+h_l)=\prod_{l=0}^{T-1}(1+h_l)$, $n\in\mathbb{N}$, we have $E_1(x,y)_n=(x_n,y_n)$ for all $n\in\mathbb{N}$. The equality $E_2(x,y)_n=(x_n,y_n)$ for $n\in\mathbb{N}$ can be obtained by using a similar procedure. The proof is complete. □

In preparation for the next result we assume that there exist positive constants W_1, W_2, K_1, and K_2 such that

$$|f(x)|\le W_1 \tag{2.4}$$

$$|g(y)|\le W_2 \tag{2.5}$$

$$|\alpha_h|\sum_{i=n}^{n+T-1}\left|\prod_{l=i+1}^{n+T-1}(1+h_l)\right|\sum_{m=-\infty}^{i}|a_{i,m}|\le K_1 \tag{2.6}$$

$$|\alpha_p|\sum_{i=n}^{n+T-1}\left|\prod_{l=i+1}^{n+T-1}(1+p_l)\right|\sum_{m=-\infty}^{i}|b_{i,m}|\le K_2 \tag{2.7}$$

for all $n\in\mathbb{Z}$ and all $(x,y)\in\Omega(W)$.

Theorem 2. *In addition to the assumptions of Lemma 1 suppose that (2.4-2.7) hold. Then (1.1) has a T-periodic solution.*

Proof. From Lemma 1, we can deduce that $E(x,y)_{n+T}=E(x,y)_n$ for all $n\in\mathbb{N}$ and any $(x,y)\in\Omega(W)$. Moreover, if $(x,y)\in\Omega(W)$ then

$$|E_1(x,y)_n|\le|\alpha_h|\sum_{i=n}^{n+T-1}\left|\prod_{l=i+1}^{n+T-1}(1+h_l)\right|\sum_{m=-\infty}^{i}|a_{i,m}|\,|f(y_m)|\le W_1K_1, \tag{2.8}$$

and

$$|E_2(x,y)_n|\le|\alpha_p|\sum_{i=n}^{n+T-1}\left|\alpha_p\prod_{l=i+1}^{n+T-1}(1+p_l)\right|\sum_{m=-\infty}^{i}|b_{i,m}|\,|g(x_m)|\le W_2K_2 \tag{2.9}$$

for all $n \in \mathbb{N}$. If we set $W = \max\{W_1K_1, W_2K_2\}$ then E maps $\Omega(W)$ into itself. Now we show that E is continuous. Let $\{(x^l, y^l)\}$, $l \in \mathbb{N}$, be a sequence in $\Omega(W)$ such that

$$\lim_{l\to\infty} \left\|(x^l, y^l) - (x, y)\right\| = \lim_{l\to\infty} \left(\max_{n\in[1,T]_{\mathbb{Z}}} \left\{ \left|x_n^l - x_n\right|, \left|y_n^l - y_n\right| \right\} \right) \\ = 0.$$

Since $\Omega(W)$ is closed, we must have $(x, y) \in \Omega(W)$. Then by definition of E we have

$$\left\|E(x^l, y^l) - E(x, y)\right\| = \max\left\{ \max_{n\in[1,T]_{\mathbb{Z}}} \left|E_1(x^l, y^l)_n - E_1(x, y)_n\right|, \right. \\ \left. \max_{n\in[1,T]_{\mathbb{Z}}} \left|E_2(x^l, y^l)_n - E_2(x, y)_n\right| \right\},$$

in which

$$\left|E_1(x^l, y^l)_n - E_1(x, y)_n\right| = |\alpha_h| \left| \sum_{i=n}^{n+T-1} \left(\prod_{l=i+1}^{n+T-1} (1 + h_l) \right) \sum_{m=-\infty}^{i} a_{i,m} f(y_m^l) - \right. \\ \left. \sum_{i=n}^{n+T-1} \left(\prod_{l=i+1}^{n+T-1} (1 + h_l) \right) \sum_{m=-\infty}^{i} a_{i,m} f(y_m) \right| \\ \leq |\alpha_h| \sum_{i=n}^{n+T-1} \left| \prod_{l=i+1}^{n+T-1} (1 + h_l) \right| \sum_{m=-\infty}^{i} |a_{i,m}| \left| f(y_m^l) - f(y_m) \right|.$$

Similarly,

$$\left|E_2(x^l, y^l)_n - E_2(x, y)_n\right| \leq |\alpha_p| \sum_{i=n}^{n+T-1} \left| \prod_{l=i+1}^{n+T-1} (1 + p_l) \right| \sum_{m=-\infty}^{i} |b_{i,m}| \left| g(x_m^l) - g(x_m) \right|.$$

The continuity of f and g along with the Lebesgue dominated convergence theorem imply that

$$\lim_{l\to\infty} \left\|E(x^l, y^l) - E(x, y)\right\| = 0.$$

This shows that E is continuous. Finally, we have to show that $E\Omega(W)$ is precompact. Let $\{(x^l, y^l)\}_{l\in\mathbb{N}}$ be a sequence in $\Omega(W)$. For each fixed $l \in \mathbb{N}$, $\{(x_n^l, y_n^l)\}_{n\in\mathbb{Z}}$ is a bounded sequence of real pairs. Then by *Bolzano-Weierstrass Theorem,* $\{(x_n^l, y_n^l)\}_{n\in\mathbb{Z}}$ has a convergent subsequence $\{(x_{n_k}^l, y_{n_k}^l)\}$. By repeating the diagonalization process for each $l \in \mathbb{N}$, we can construct a convergent subsequence $\{(x^{l_k}, y^{l_k})\}_{l_k\in\mathbb{N}}$ of $\{(x^l, y^l)\}_{l\in\mathbb{N}}$ in $\Omega(W)$. Since E is continuous, we deduce that $\{E(x^l, y^l)\}_{l\in\mathbb{N}}$ has a convergent subsequence in $E\Omega(W)$. This means, $E\Omega(W)$ is precompact. By Schauder's fixed point theorem we conclude that there exists a pair $(x, y) \in \Omega(W)$ such that $E(x, y) = (x, y)$. $\square$

Theorem 3. *In addition to the assumptions of Lemma 1, we assume that (2.4), (2.6) and (2.7) hold. If g is a non-decreasing function satisfying*

$$|g(x)| \leq g(|x|), \tag{2.10}$$

then (1.1) has a T-periodic solution.

Proof. By (2.6) and (2.8) we already have

$$|E_1(x, y)| \leq W_1K_1 \text{ for all } (x, y) \in \Omega(W).$$

This along with (2.10) imply

$$
\begin{aligned}
|E_2(x,y)_n| &\leq \sum_{i=n}^{n+T-1} \left| \alpha_p \prod_{l=i+1}^{n+T-1} (1+p_l) \right| \sum_{m=-\infty}^{i} |b_{i,m}| \, |g(x_m)| \\
&\leq \sum_{i=n}^{n+T-1} \left| \alpha_p \prod_{l=i+1}^{n+T-1} (1+p_l) \right| \sum_{m=-\infty}^{i} |b_{i,m}| \, g(|E_1(x,y)|) \\
&\leq K_2 g(W_1 K_1).
\end{aligned}
$$

If we set $W = \max\{W_1K_1, K_2g(W_1K_1)\}$, then the rest of the proof is similar to the proof of Theorem 2 and hence we omit it. □

Similarly, we can give the following result.

Theorem 4. *In addition to the assumptions of Lemma 1, we assume (2.5), (2.6) and (2.7) hold. If f is a non-decreasing function satisfying*

$$|f(y)| \leq f(|y|),$$

then (1.1) has a T-periodic solution.

Example 1. *Let*

$$
\begin{aligned}
h_n &= 1 + \cos n\pi, \\
p_n &= 1 - \cos n\pi, \\
a_{n,i} &= b_{n,i} = e^{i-n},
\end{aligned}
$$

and

$$f(x) = \sin x \text{ and } g(x) = \sin 2x.$$

Then (1.1) turns into the following system

$$
\begin{cases}
\Delta x_n = (1+\cos n\pi)x_n + \sum\limits_{i=-\infty}^{n} e^{i-n} \sin(y_i), \\
\Delta y_n = (1-\cos n\pi)y_n + \sum\limits_{i=-\infty}^{n} e^{i-n} \sin(2x_i)
\end{cases}.
$$

It can be easily verified that conditions (1.2-2.3) and (2.4-2.7) hold. By Theorem 2, there exists a 2-periodic solution $(x,y) = \{(x_n,y_n)\}_{ne\mathbb{Z}}$ of system (1.1) satisfying

$$
\begin{aligned}
x_n &= -\frac{1}{2} \sum_{i=n}^{n+1} \prod_{l=i+1}^{n+1} (2+\cos(l\pi)) \sum_{m=-\infty}^{i} e^{m-i} \sin(y_m), \\
y_n &= -\frac{1}{2} \sum_{i=n}^{n+1} \prod_{l=i+1}^{n+1} (2-\cos(l\pi)) \sum_{m=-\infty}^{i} e^{m-i} \sin(2x_m),
\end{aligned}
$$

for all $n \in \mathbb{N}$.

3. Asymptotic Periodicity

In this section, we are going to show the existence of an asymptotically T-periodic solution of system (1.1) by using Schauder's fixed point theorem. First we state the following definition.

Definition 1. *A sequence $\{x_n\}_{n\in\mathbb{Z}}$ is called asymptotically T-periodic if there exist two sequences u_n and v_n such that u_n is T-periodic, $\lim_{n\to\infty} v_n = 0$, and $x_n = u_n + v_n$ for all $n \in \mathbb{Z}$.*

First, we suppose that

$$\prod_{j=0}^{T-1}(1+h_j)=1 \text{ and } \prod_{j=0}^{T-1}(1+p_j)=1. \tag{3.1}$$

Then we define the sequences $\varphi:=\{\varphi_n\}_{n\in\mathbb{N}}$ and $\psi:=\{\psi_n\}_{n\in\mathbb{N}}$ as follows

$$\varphi_n:=\prod_{j=0}^{n-1}\frac{1}{1+h_j} \text{ and } \psi_n:=\prod_{j=0}^{n-1}\frac{1}{1+p_j}. \tag{3.2}$$

Furthermore, we define the constants m_k, M_k, $k=1,2$, by

$$m_1:=\min_{i\in[1,T]_{\mathbb{Z}}}|\varphi_i|\,,\quad M_1:=\max_{i\in[1,T]_{\mathbb{Z}}}|\varphi_i|\,,\ m_2:=\min_{i\in[1,T]_{\mathbb{Z}}}|\psi_i|\,,\quad M_2:=\max_{i\in[1,T]_{\mathbb{Z}}}|\psi_i|\,.$$

We note that in this section, we do not assume (1.3-1.4) but instead we ask that the series

$$\sum_{i=0}^{\infty}\sum_{m=-\infty}^{i}|a_{i,m}|<\infty \text{ and } \sum_{i=0}^{\infty}\sum_{m=-\infty}^{i}|b_{i,m}|<\infty \tag{3.3}$$

converge to a and b, respectively. Observe that (3.3) implies

$$\lim_{n\to\infty}\sum_{i=n}^{\infty}\sum_{m=-\infty}^{i}|a_{i,m}|=\lim_{n\to\infty}\sum_{i=n}^{\infty}\sum_{m=-\infty}^{i}|b_{i,m}|=0. \tag{3.4}$$

Theorem 5. *Suppose that (2.4-2.5), (3.1), and (3.3-3.4) hold. Then system (1.1) has an asymptotically T-periodic solution $(x,y)=\{(x_n,y_n)\}_{n\in\mathbb{Z}}$ satisfying*

$$x_n:=u_n^{(1)}+v_n^{(1)}$$
$$y_n:=u_n^{(2)}+v_n^{(2)}$$

for $n\in\mathbb{N}$, where

$$u_n^{(1)}=c_1\prod_{j=0}^{n-1}(1+h_j),\qquad u_n^{(2)}=c_2\prod_{j=0}^{n-1}(1+p_j),\ n\in\mathbb{Z}^+$$

c_1 and c_2 are positive constants, and

$$\lim_{n\to\infty}v_n^{(1)}=\lim_{n\to\infty}v_n^{(2)}=0.$$

Proof. Due to the T-periodicity of the sequences $\{h_n\}_{n\in\mathbb{Z}}$ and $\{p_n\}_{n\in\mathbb{Z}}$ and by (3.1-3.2) we have

$$\varphi_n\in\{\varphi_1,\varphi_2,...,\varphi_T\} \text{ and } \psi_n\in\{\psi_1,\psi_2,...,\psi_T\}$$

for all $n\in\mathbb{N}$. This means

$$m_1\le|\varphi_n|\le M_1 \tag{3.5}$$
$$m_2\le|\psi_n|\le M_2 \tag{3.6}$$

for all $n\in\mathbb{N}$. Let B be the set of all real bounded sequences $x=\{x_n\}_{ne\mathbb{Z}}$. Denote by $\mathbb{B}$ the Banach space of all pairs of real bounded sequences $(x,y)=\{(x_n,y_n)\}_{n\in\mathbb{Z}}$, $x,y\in B$, endowed with the maximum norm

$$\|(x,y)\|=\max\{\sup_{n\in\mathbb{Z}}|x_n|\,,\sup_{n\in\mathbb{Z}}|y_n|\}.$$

For a positive constant W^* we define

$$\Omega^*(W^*):=\{(x,y)\in\mathbb{B}:\|(x,y)\|\le W^*\}.$$

Then, $\Omega^*(W^*)$ is a nonempty bounded convex, and closed subset of $\mathbb{B}$. For any pair $(x,y) = \{(x_n(\eta), y_n(\zeta))\}_{n\in\mathbb{Z}} \in \Omega^*(W^*)$ with an initial sequence $\{(\eta_n, \zeta_n)\}_{n\in\mathbb{Z}^-}$ in S define the mapping E^* on $\Omega^*(W^*)$ by

$$E^*(x,y) = \{E^*(x,y)_n\}_{n\in\mathbb{Z}} = \left\{ \begin{pmatrix} E_1^*(x,y)_n \\ E_2^*(x,y)_n \end{pmatrix} \right\}_{n\in\mathbb{Z}},$$

where

$$E_1^*(x,y)_n := \begin{cases} \eta_n & \text{for } n \in \mathbb{Z}^- \\ c_1 \frac{1}{\varphi_n} - \sum\limits_{i=n}^{\infty} \sum\limits_{m=-\infty}^{i} \frac{\varphi_{i+1}}{\varphi_n} a_{i,m} f(y_m) & \text{for } n \in \mathbb{N} \end{cases}, \tag{3.7}$$

and

$$E_2^*(x,y)_n := \begin{cases} \zeta_n & \text{for } n \in \mathbb{Z}^- \\ c_2 \frac{1}{\psi_n} - \sum\limits_{i=n}^{\infty} \sum\limits_{m=-\infty}^{i} \frac{\psi_{i+1}}{\psi_n} b_{i,m} g(x_m) & \text{for } n \in \mathbb{N} \end{cases}. \tag{3.8}$$

We will show that the mapping E^* has a fixed point in $\mathbb{B}$. First, we demonstrate that $E^*\Omega^*(W^*) \subset \Omega^*(W^*)$. If $(x,y) \in \Omega^*(W^*)$, then

$$\left| E_1^*(x,y)_n - c_1 \frac{1}{\varphi_n} \right| \le M_1 m_1^{-1} W_1 \sum_{i=n}^{\infty} \sum_{m=-\infty}^{i} |a_{i,m}| \tag{3.9}$$

$$\le M_1 m_1^{-1} W_1 \sum_{i=0}^{\infty} \sum_{m=-\infty}^{i} |a_{i,m}|$$

$$= M_1 m_1^{-1} W_1 a, \tag{3.10}$$

and

$$\left| E_2^*(x,y)_n - c_2 \frac{1}{\psi_n} \right| \le M_2 m_2^{-1} W_2 \sum_{i=n}^{\infty} \sum_{m=-\infty}^{i} |b_{i,m}| \tag{3.11}$$

$$\le M_2 m_2^{-1} W_2 \sum_{i=0}^{\infty} \sum_{m=-\infty}^{i} |b_{i,m}|$$

$$= M_2 m_2^{-1} W_2 b \tag{3.12}$$

for all $n \in \mathbb{N}$. This implies that

$$|E_1^*(x,y)_n| \le M_1 m_1^{-1} W_1 a + \frac{c_1}{m_1}$$

and

$$|E_2^*(x,y)_n| \le M_2 m_2^{-1} W_2 b + \frac{c_2}{m_2}$$

for all $n \in \mathbb{N}$. If we set

$$W^* = \max\{M_1 m_1^{-1} W_1 a + \frac{c_1}{m_1}, M_2 m_2^{-1} W_2 b + \frac{c_2}{m_2}\},$$

then we have $E^*\Omega^*(W^*) \subset \Omega^*(W^*)$ as desired.

Next, we show that E^* is continuous. Let $\{(x^q, y^q)\}_{q\in\mathbb{N}}$ be a sequence in $\Omega^*(W^*)$ such that $\lim_{q\to\infty} \|(x^q, y^q) - (x,y)\| = 0$, where $(x,y) = \{(x_n, y_n)\}_{n\in\mathbb{Z}}$. Since $\Omega^*(W^*)$ is closed, we must have $(x,y) \in \Omega^*(W^*)$. From (3.7) and (3.8), we have

$$|E_1^*(x^q, y^q)_n - E_1^*(x,y)_n| \le \sum_{i=n}^{\infty} \sum_{m=-\infty}^{i} \left| \frac{\varphi_{i+1}}{\varphi_n} \right| |a_{i,m}| \, |f(y_m^q) - f(y_m)|$$

and

$$|E_2^*(x^q,y^q)_n - E_2^*(x,y)_n| \le \sum_{i=n}^{\infty}\sum_{m=-\infty}^{i}\left|\frac{\psi_{i+1}}{\psi_n}\right| |b_{i,m}|\, |g(x_m^q) - g(x_m)|$$

for all $n \in \mathbb{N}$. Since f and g are continuous, we have by the Lebesgue dominated convergence theorem that

$$\lim_{q\to\infty} \|E^*(x^q,y^q) - E^*(x,y)\| = 0.$$

As we did in the proof of Theorem 2 we can show that E^* has a fixed point in $\Omega^*(W^*)$. On the other hand, using a similar procedure that we have employed in the proof of Lemma 1, we can deduce that any solution $(x,y) = \{(x_n,y_n)\}_{n\in\mathbb{Z}}$ of the system (1.1) is a fixed point for the operator E^*. This means $E^*(x,y) = (x,y)$ or equivalently,

$$x_n = c_1\frac{1}{\varphi_n} - \sum_{i=n}^{\infty}\sum_{m=-\infty}^{i}\frac{\varphi_{i+1}}{\varphi_n}a_{i,m}f(y_m) \tag{3.13}$$

and

$$y_n = c_2\frac{1}{\psi_n} - \sum_{i=n}^{\infty}\sum_{m=-\infty}^{i}\frac{\psi_{i+1}}{\psi_n}b_{i,m}g(x_m). \tag{3.14}$$

Conversely, any pair $(x,y) = \{(x_n,y_n)\}_{n\in\mathbb{Z}}$ satisfying (3.13) and (3.14) will also satisfy

$$\begin{aligned} x_{n+1} - x_n(1+h_n) &= c_1\Big(\prod_{j=0}^{n}(1+h_j) - (1+h_n)\prod_{j=0}^{n-1}(1+h_j)\Big) \\ &\quad + (1+h_n)\sum_{i=n}^{\infty}\sum_{m=-\infty}^{i}\frac{\varphi_{i+1}}{\varphi_n}a_{i,m}f(y_m) \\ &\quad - \sum_{i=n+1}^{\infty}\sum_{m=-\infty}^{i}\frac{\varphi_{i+1}}{\varphi_{n+1}}a_{i,m}f(y_m), \end{aligned}$$

and hence,

$$\begin{aligned} x_{n+1} - x_n(1+h_n) &= \sum_{i=n}^{\infty}\sum_{m=-\infty}^{i}\frac{(1+h_n)\prod_{j=0}^{n-1}(1+h_j)}{\prod_{j=0}^{i}(1+h_j)}a_{i,m}f(y_m) \\ &\quad - \sum_{i=n+1}^{\infty}\sum_{m=-\infty}^{i}\frac{\prod_{j=0}^{n}(1+h_j)}{\prod_{j=0}^{i}(1+h_j)}a_{i,m}f(y_m) \\ &= \sum_{m=-\infty}^{n} a_{n,m}f(y_m). \end{aligned}$$

That is, any fixed point $(x,y) = \{(x_n,y_n)\}_{n\in\mathbb{Z}}$ of the operator E^* satisfies the first equation in (1.1). Similarly, one may show that the second equation holds.
For an arbitrary fixed point $(x,y) \in \Omega^*(W^*)$ of E^*, we have

$$\lim_{n\to\infty}\left|x_n - c_1\frac{1}{\varphi_n}\right| = \lim_{n\to\infty}\left|E_1^*(x,y)_n - c_1\frac{1}{\varphi_n}\right| = 0 \tag{3.15}$$

and

$$\lim_{n\to\infty}\left|y_n - c_2\frac{1}{\psi_n}\right| = \lim_{n\to\infty}\left|E_2(x,y)_n - c_2\frac{1}{\psi_n}\right| = 0. \tag{3.16}$$

Choosing

$$u_n^{(1)} = c_1\frac{1}{\varphi_n}, \quad v_n^{(1)} = -\sum_{i=n}^{\infty}\sum_{m=-\infty}^{i}\frac{\varphi_{i+1}}{\varphi_n}a_{i,m}f(y_m) \tag{3.17}$$

and

$$u_n^{(2)} = c_2\frac{1}{\psi_n}, \quad v_n^{(2)} = -\sum_{i=n}^{\infty}\sum_{m=-\infty}^{i}\frac{\psi_{i+1}}{\psi_n}b_{i,m}g(x_m), \tag{3.18}$$

we have $x_n = u_n^{(1)} + v_n^{(1)}$ and $y_n = u_n^{(2)} + v_n^{(2)}$. By (3.15) and (3.16), $v_n^{(1)}$ and $v_n^{(2)}$ tend to 0 when $n\to\infty$. Left to show that $u_n^{(1)}$ and $u_n^{(2)}$ are T-periodic.

$$\begin{aligned}
u_{n+T}^{(1)} = c_1\prod_{j=0}^{n+T-1}(1+h_j) &= c_1\prod_{j=0}^{n-1}(1+h_j)\prod_{j=n}^{n+T-1}(1+h_j) \\
&= c_1\prod_{j=0}^{n-1}(1+h_j)\prod_{j=0}^{T-1}(1+h_j) \\
&= c_1\prod_{j=0}^{n-1}(1+h_j), \text{ by (3.1).}
\end{aligned}$$

Proof for $u_n^{(2)}$ is identical. □

Example 2. *Consider the system (1.1) with the following entries*

$$\begin{aligned}
h_n = p_n &= \begin{cases} 1, & \text{if } n = 2k+1 \text{ for } k\in\mathbb{Z} \\ -\frac{1}{2}, & \text{if } n = 2k \text{ for } k\in\mathbb{Z} \end{cases}, \\
a_{n,i} &= e^{i-2n}, \text{ for } n,i\in\mathbb{Z} \\
b_{n,i} &= e^{2i-3n}, \text{ for } n,i\in\mathbb{Z} \\
f(x) &= \cos x \text{ and } g(x) = \cos 2x.
\end{aligned}$$

Then (1.1) turns into the following system:

$$\begin{cases} \Delta x_n = h_n x_n + \sum\limits_{i=-\infty}^{n} e^{i-2n}\cos(y_i), \\ \Delta y_n = p_n y_n + \sum\limits_{i=-\infty}^{n} e^{2i-3n}\cos(2x_i) \end{cases}.$$

Obviously, the sequences $\{h_n\}_{n\in\mathbb{Z}}$ and $\{p_n\}_{n\in\mathbb{Z}}$ are 2−periodic and all conditions of Theorem 5 are satisfied. Hence, we conclude by Theorem 5 the existence of an asymptotically 2−periodic solution $(x,y) = \{(x_n,y_n)\}_{n\in\mathbb{Z}}$ satisfying

$$\begin{aligned}
x_n &= c_1\frac{1}{\varphi_n} - \sum_{i=n}^{\infty}\sum_{m=-\infty}^{i}\frac{\varphi_{i+1}}{\varphi_n}e^{m-2i}\cos(y_m) \\
y_n &= c_2\frac{1}{\psi_n} - \sum_{i=n}^{\infty}\sum_{m=-\infty}^{i}\frac{\psi_{i+1}}{\psi_n}e^{2m-3i}\cos(2x_m),
\end{aligned}$$

for all $n \in \mathbb{N}$, where c_1 and c_2 are positive constants, $\varphi := \{\varphi_n\}_{n\in\mathbb{N}}$ and $\psi := \{\psi_n\}_{n\in\mathbb{N}}$ are as in (3.2).

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